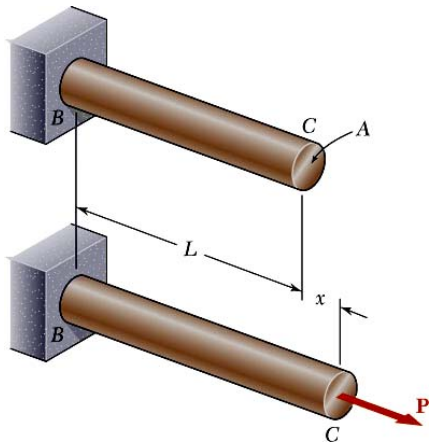
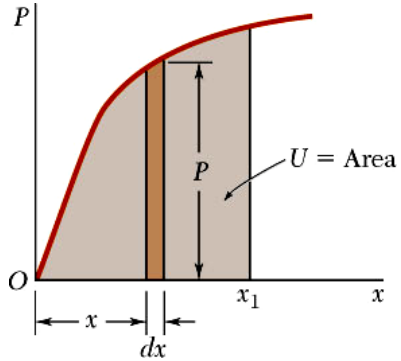
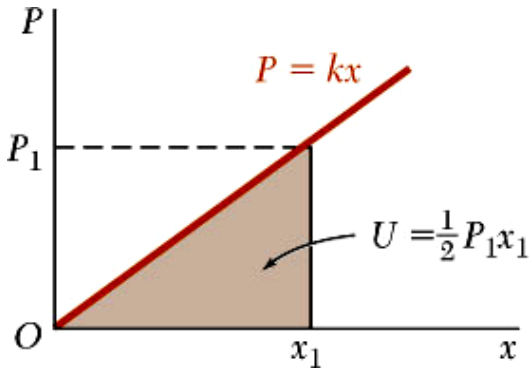
ENERJİ METOTLARI: Eksenel Yüklemede Şekil değiştirme Enerjisi







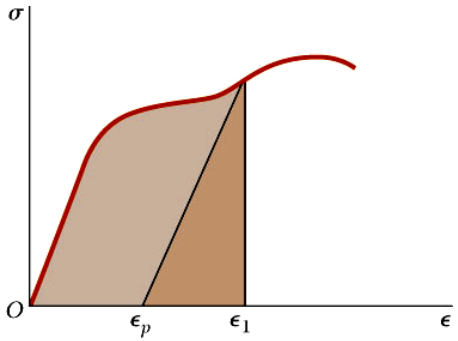
- ***dx*** uzaması için yapılan iş: $dU = P\,dx = elementary\ work$

- x_1 uzaması için yapılan toplam iş: $U = \int_0^{x_1} P\,dx = total\ work = strain\ energy$

- Lineer elastic deformasyon durumunda, $U = \int_0^{x_1} kx\,dx = \frac{1}{2}kx_1^2 = \frac{1}{2}P_1x_1$

- Boyut etkisini ortadan kaldırmak için birim hacme karşılık gelen şekil değiştirme enerjisi

$$\frac{U}{V} = \int_0^{x_1} \frac{P}{A}\frac{dx}{L} \rightarrow \boxed{u = \int_0^{\varepsilon_1} \sigma_x\, d\varepsilon_x}$$



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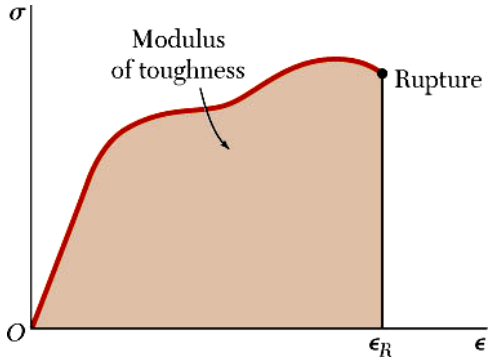

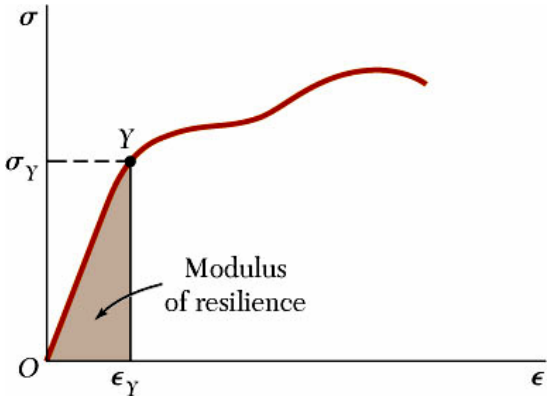

$\varepsilon_1 = \varepsilon_R$ ise şekil değiştirme enerjisi (eğrinin altındaki alan) Tokluk Modülü olarak adlandırılır.

- Eğer gerilme orantı limitinin içinde (lastik bölgede) kalıyorsa herhangi bir ε_1 değeri için birim şekil değiştirme enerjisi:

$$u = \int_0^{\varepsilon_1} E\varepsilon_x \, d\varepsilon_x = \frac{E\varepsilon_1^2}{2} = \frac{\sigma_1^2}{2E}$$

$$u_Y = \frac{\sigma_Y^2}{2E} = modulus\ of\ resilience$$

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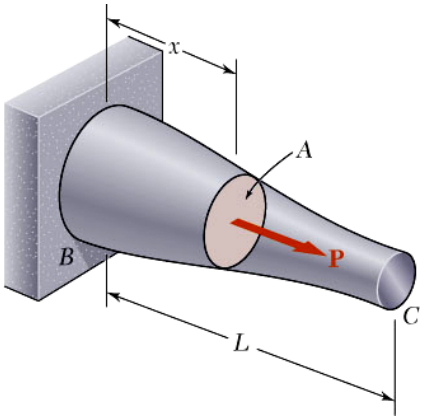
Normal Gerilmeler için Elastik şekil değiştirme enerjisi

- ***Eksenel yükleme:***



- Değişken kesitli çubuklar (gerilme dağılımı düzgün değil) için toplam şekil değiştirme enerjisi,

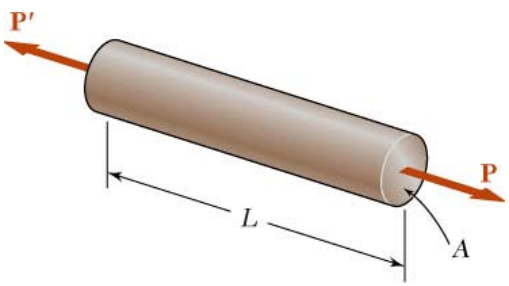
$$u = \lim_{\Delta V \to 0} \frac{\Delta U}{\Delta V} = \frac{dU}{dV} \qquad U = \int u \, dV$$

- Elastik Bölgede; $U = \int \frac{\sigma_x^2}{2E} dV = elastic\ strain\ energy$

- Eksenel yüklemede:

$$\sigma_x = P/A \qquad dV = A\,dx$$

$$\boxed{U = \int_0^L \frac{P^2}{2AE} dx}$$



- ***Ör:*** Düzgün kesitli bir çubuk için;

$$U = \frac{P^2 L}{2AE}$$

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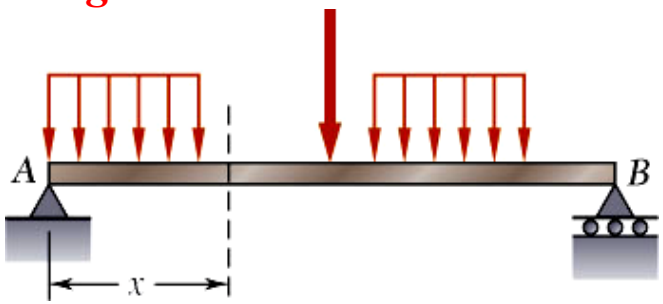
- ***Eğilme:***



$$\sigma_x = \frac{My}{I}$$

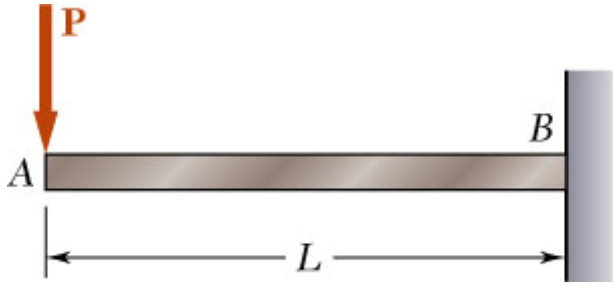
- Eğilmeye maruz bir kiriş için,

$$U = \int \frac{\sigma_x^2}{2E} dV = \int \frac{M^2 y^2}{2EI^2} dV$$

- $dV = dA\ dx,$

$$U = \int_0^L \int_A \frac{M^2 y^2}{2EI^2} dA\, dx = \int_0^L \frac{M^2}{2EI^2} \left(\int_A y^2 dA \right) dx$$

$$\boxed{\boldsymbol{U} = \int_0^L \frac{M^2}{2EI} dx}$$



- ***Ör:*** Serbest Uçtan P düşey yüküne maruz ankastre kiriş:

$$M = -Px$$

$$U = \int_0^L \frac{P^2 x^2}{2EI} dx = \frac{P^2 L^3}{6EI}$$

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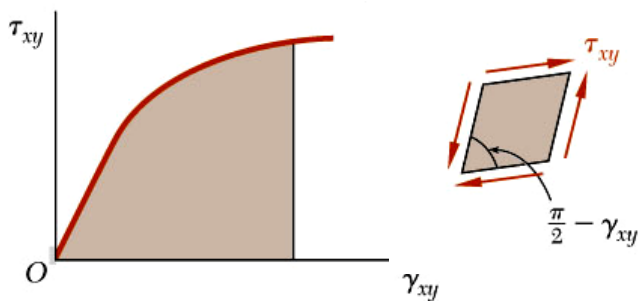
Kayma Gerilmeleri için Elastik şekil değiştirme enerjisi

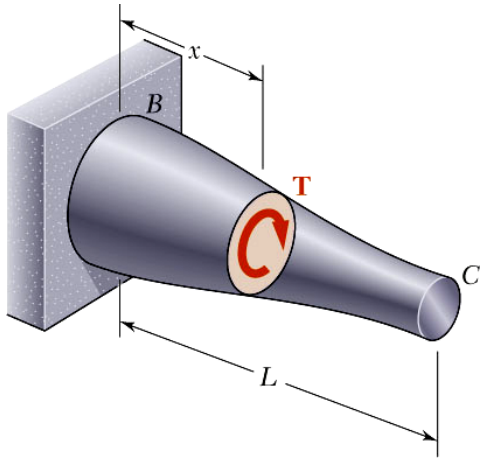


τ_{xy} Elastik bölgede ise;

$$u = \int_0^{\gamma_{xy}} \tau_{xy}\, d\gamma_{xy} \qquad u = \tfrac{1}{2} G \gamma_{xy}^2 = \tfrac{1}{2} \tau_{xy} \gamma_{xy} = \frac{\tau_{xy}^2}{2G}$$

- Toplam şekil değiştirme enerjisi;

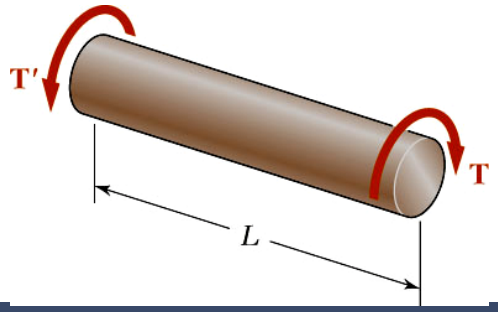
$$\boxed{U = \int u\, dV = \int \frac{\tau_{xy}^2}{2G}\, dV}$$



$$U = \int \frac{\tau_{xy}^2}{2G}\, dV = \int \frac{T^2 \rho^2}{2GJ^2}\, dV; \qquad dV = dA\, dx,$$

$$U = \int_0^L \int_A \frac{T^2 \rho^2}{2GJ^2}\, dA\, dx = \int_0^L \frac{T^2}{2GJ^2} \left(\int_A \rho^2 dA \right) dx$$

$$\boxed{U = \int_0^L \frac{T^2}{2GJ}\, dx}$$



- ***Ör:*** Düzgün kesitli bir çubuk için; $U = \dfrac{T^2 L}{2GJ}$

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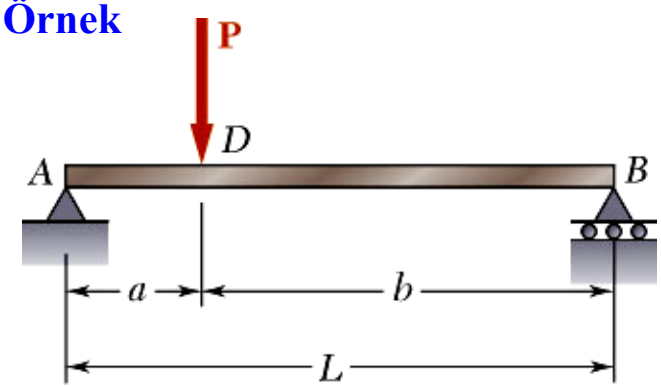
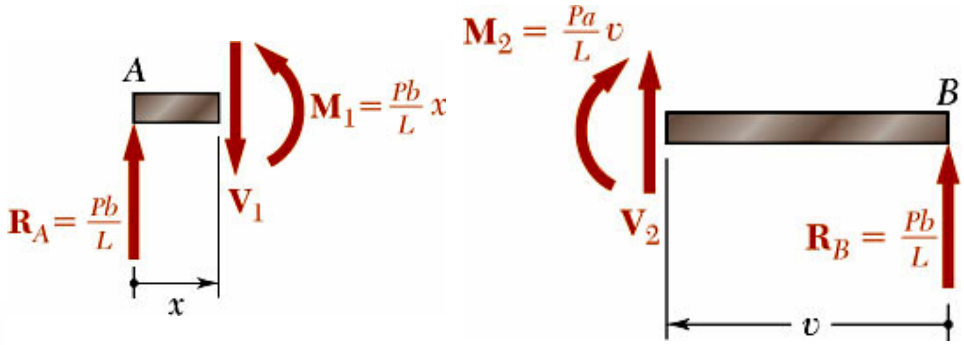
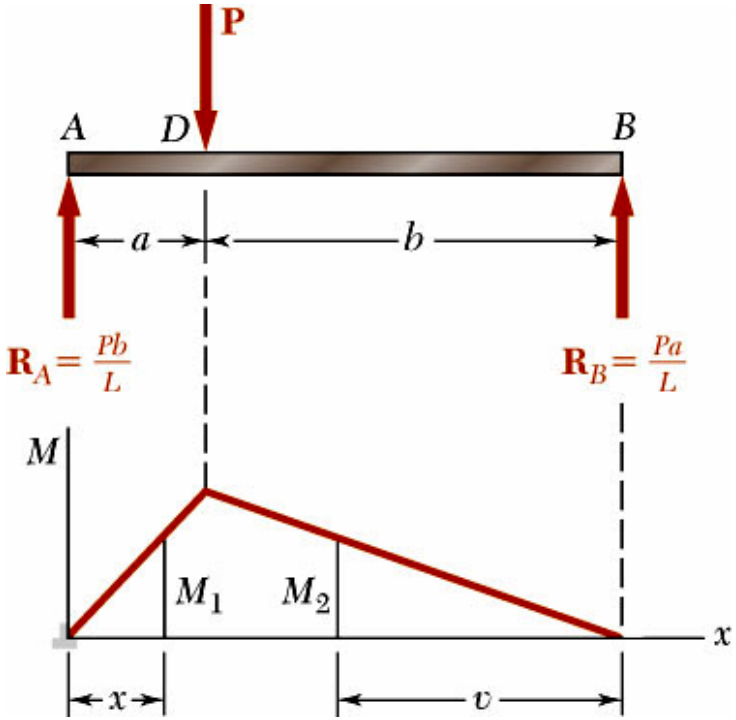
Örnek

Sadece eğilmedeki normal gerilmeleri göz önüne alarak, şekildeki yükleme durumu için kirişteki şekil değiştirme enerjisini bulunuz







$$M_1 = \frac{Pb}{L}x \qquad M_2 = \frac{Pa}{L}v$$

$$U = \int_0^a \frac{M_1^2}{2EI}dx + \int_0^b \frac{M_2^2}{2EI}dv$$

$$= \frac{1}{2EI}\int_0^a \left(\frac{Pb}{L}x\right)^2 dx + \frac{1}{2EI}\int_0^b \left(\frac{Pa}{L}x\right)^2 dx$$

$$= \frac{1}{2EI}\frac{P^2}{L^2}\left(\frac{b^2a^3}{3} + \frac{a^2b^3}{3}\right) = \frac{P^2a^2b^2}{6EIL^2}(a+b)$$

$$\boxed{U = \frac{P^2a^2b^2}{6EIL}}$$

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Genel Yükleme (Gerilme) durumu için şekil değiştirme enerjisi

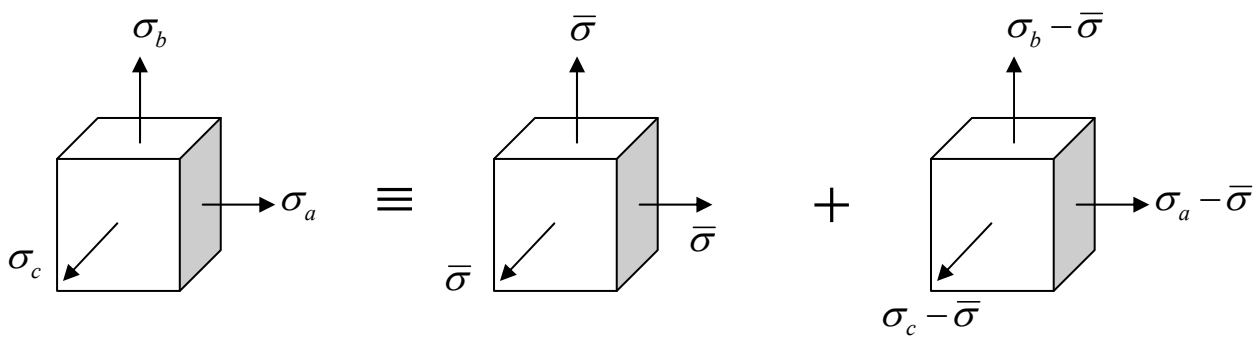
- Genel yükleme durumu için birim şekil değiştirme enerjisi;

$$u = \tfrac{1}{2}\left(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx}\right) \longleftarrow \text{Hooke Bağıntıları}$$

$$u = \frac{1}{2E}\left[\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - 2\nu\left(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x\right)\right] + \frac{1}{2G}\left[\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2\right]$$

- Elastik, izotropik bir malzemede asal gerilmeler cinsinden birim şekil değiştirme enerjisi;

$$u = \frac{1}{2E}\left[\sigma_a^2 + \sigma_b^2 + \sigma_c^2 - 2\nu\left(\sigma_a\sigma_b + \sigma_b\sigma_c + \sigma_c\sigma_a\right)\right]$$



$$\varepsilon_v = 0$$

$$\varepsilon_v = \varepsilon_a + \varepsilon_b + \varepsilon_c = 0$$

$$\bar{\sigma} = \frac{\sigma_a + \sigma_b + \sigma_c}{3}$$

Hacim değişikliği + çarpılma

Sadece hacim değişikliği var (çarpılma yok)

Sadece çarpılma var (hacim değişikliği yok $\varepsilon_v = 0$)

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$$u = \frac{1}{2E}\left[\sigma_a^2 + \sigma_b^2 + \sigma_c^2 - 2\nu\left(\sigma_a\sigma_b + \sigma_b\sigma_c + \sigma_c\sigma_a\right)\right] = u_v + u_d$$

$$G = \frac{E}{2(1+\upsilon)}$$

$$u_v = \frac{1-2\nu}{6E}\left(\sigma_a + \sigma_b + \sigma_c\right)^2 = \textit{due to volume change}$$

$$u_d = \frac{1}{12G}\left[\left(\sigma_a - \sigma_b\right)^2 + \left(\sigma_b - \sigma_c\right)^2 + \left(\sigma_c - \sigma_a\right)^2\right] = \textit{due to distortion}$$

- İki boyutlu durum (düzlem gerilme) için çarpılma şekil değiştirme enerjisi (σ_c=0):

$$u_d = \frac{1}{6G}\left[\sigma_a^2 - \sigma_a\sigma_b + \sigma_b^2\right]$$

- Tek eksenli gerilme durumunda ($\sigma_a = \sigma_y$, $\sigma_b = \sigma_c = 0$): $u_d = \dfrac{\sigma_y^2}{6G}$

- **Çarpılma Şekil Değiştirme Enerjisi Kriteri (Von Mises Kriteri):** $u_d < (u_d)_Y$ *olmalı*

$$\frac{1}{6G}\left[\sigma_a^2 - \sigma_a\sigma_b + \sigma_b^2\right] < \frac{\sigma_y^2}{6G}$$

$$\sigma_a^2 - \sigma_a\sigma_b + \sigma_b^2 < \sigma_y^2$$

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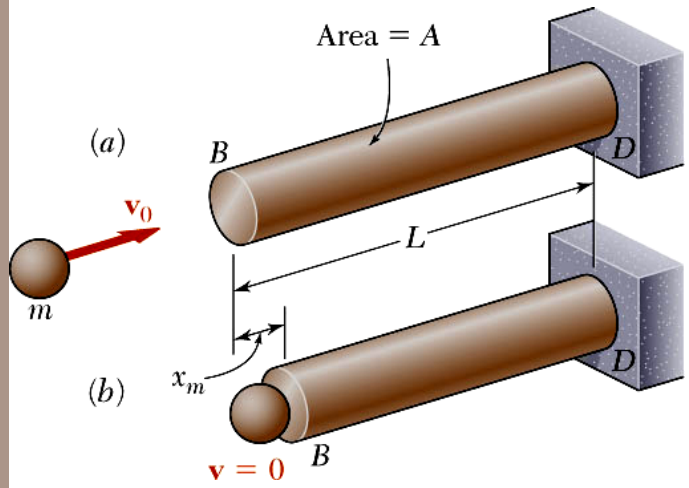
Darbeli Yükleme



- Maksimum gerilmeyi (σ_m) bulmak için, kinetik enerjinin tamamının yapıya aktarıldığı kabul edilirse:

$$T = kinetik \quad enerji$$

$$U_m = T = \tfrac{1}{2} m v_0^2$$

Dinamik çarpma için Elastik bölgedeki gerilme-şekil değiştirme diyagramının statik yükleme ile aynı olduğu kabul edilirse;

- Maksimum şekil değiştirme enerjisi;

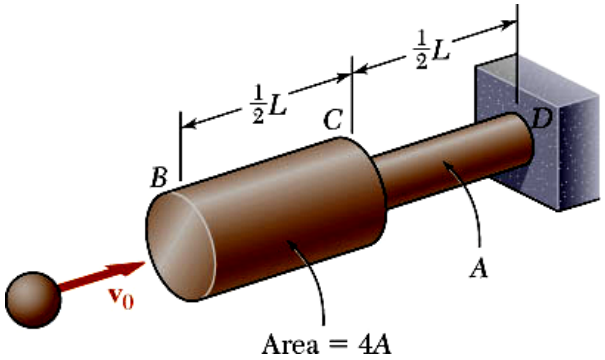
$$U_m = \int \frac{\sigma_m^2}{2E} dV$$

- **Ör:** Düzgün kesitli bir çubuk için,

$$U_m = \tfrac{1}{2} m v_0^2 = \int \frac{\sigma_m^2}{2E} dV = \frac{\sigma_m^2}{2E} V \longrightarrow \sigma_m = \sqrt{\frac{m v_0^2 E}{V}}$$

- **Örnek:**

Darbe ile aynı şekil değiştirme enerjisini meydana getiren statik kuvvet (P_m);



$$U_m = \frac{P_m^2 (L/2)}{2AE} + \frac{P_m^2 (L/2)}{2(4A)E} = \frac{5}{16} \frac{P_m^2 L}{AE} \longrightarrow P_m = \sqrt{\frac{16}{5} \frac{U_m AE}{L}}$$

Maksimum gerilme daha küçük kesitte (CD arasında) meydana gelir

$$U_m = \tfrac{1}{2} m v_0^2 ; \qquad \sigma_m = \frac{P_m}{A} = \sqrt{\frac{16}{5} \frac{U_m E}{AL}} = \sqrt{\frac{8}{5} \frac{m v_0^2 E}{AL}}$$

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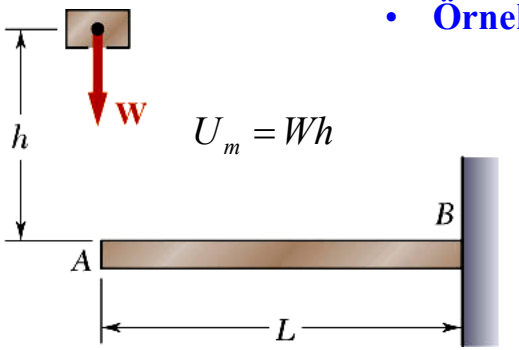

- **Örnek:** Bir ağırlığındaki bir blok h yüksekliğinden ankastre bir kirişin serbest ucuna düşürülmektedir. Buna göre kirişteki maksimum gerilmeyi bulunuz.

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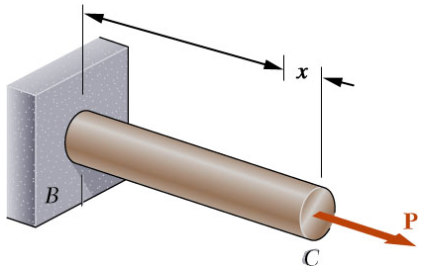
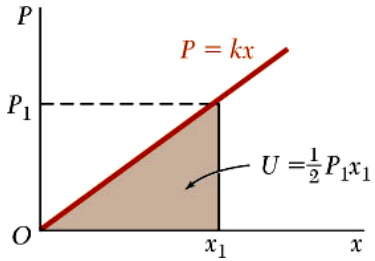
Tekil yük altında iş ve enerji

- **Eksenel Yükleme**



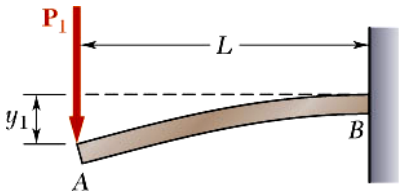


$$U = \int_0^{x_1} P\,dx = \int_0^{x_1} kx\,dx = \tfrac{1}{2}kx_1^2 = \tfrac{1}{2}P_1x_1$$

$$x_1 = \frac{P_1L}{AE} \longrightarrow U = \tfrac{1}{2}P_1\left(\frac{P_1L}{AE}\right) = \frac{P_1^2L}{2AE}$$

Not: Aynı sonuç daha önce birim şekil değiştirme enerjisinin hacimsel integrasyonu ile de bulunmuştu.

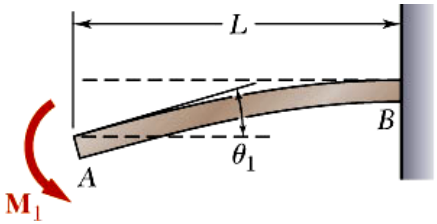
- **Transverse load**



$$U = \int_0^{y_1} P\,dy = \tfrac{1}{2}P_1y_1$$

$$= \tfrac{1}{2}P_1\left(\frac{P_1L^3}{3EI}\right) = \frac{P_1^2L^3}{6EI}$$

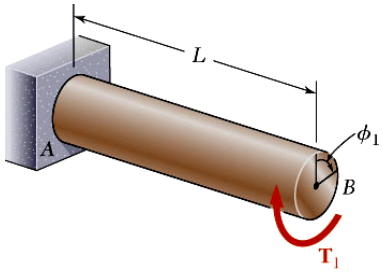
- **Bending couple**



$$U = \int_0^{\theta_1} M\,d\theta = \tfrac{1}{2}M_1\theta_1$$

$$= \tfrac{1}{2}M_1\left(\frac{M_1L}{EI}\right) = \frac{M_1^2L}{2EI}$$

- **Torsional couple**



$$U = \int_0^{\phi_1} T\,d\phi = \tfrac{1}{2}T_1\phi_1$$

$$= \tfrac{1}{2}T_1\left(\frac{T_1L}{JG}\right) = \frac{T_1^2L}{2JG}$$

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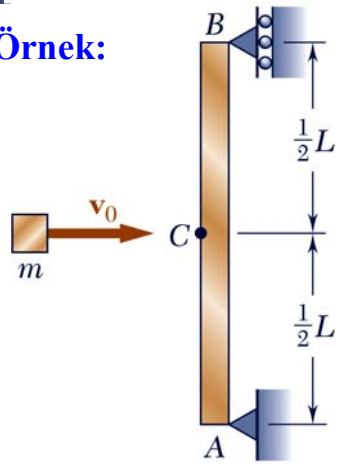
Örnek:



Şekildeki darbeli yükleme durumu için;

a) Eşdeğer statik P_m değerini bulunuz.

b) Maksimum gerilmeyi hesaplayınız.

c) C noktasındaki maksimum deplasmanı bulunuz.

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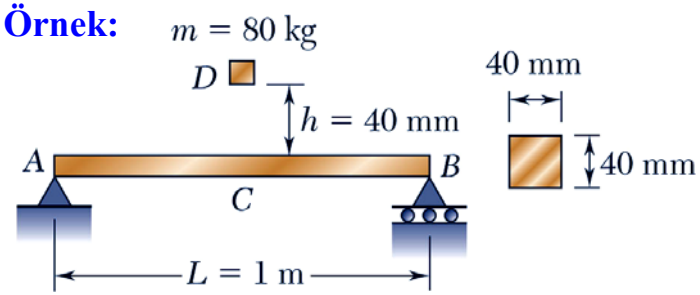
Örnek:



Şekildeki darbeli yükleme durumu için;

a)C noktasındaki maksimum deplasmanı,

b)Kirişte meydana gelen maksimum gerilmeyi bulunuz (E=70 GPa)

c) A ve B noktalarında iki yay olsaydı deplasman ve maksimum gerilme ne olurdu?

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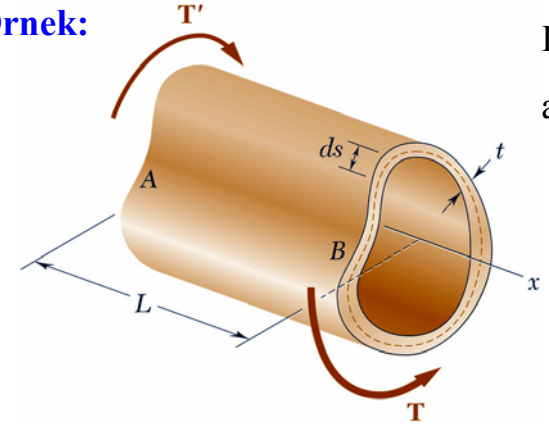
Örnek:



L boyunda içi boş ince cidarlı bir tüp için dönme açısını veren ifadeyi çıkarınız.

$$\phi = \frac{TL}{4A^2G}\oint\frac{ds}{t}$$

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"CastiglianoTeoremi" ile çözümler

- ***Castigliano teoremi*: *n*** tane kuvvete ve/veya momente maruz elastik bir cisimde bir P_j kuvvetinin uygulandığı noktadaki x_j deplasmanı (P_j ile aynı yönde) şu şekilde bulunur:

$$x_j = \frac{\partial U}{\partial P_j} \quad and \quad \theta_j = \frac{\partial U}{\partial M_j} \quad \phi_j = \frac{\partial U}{\partial T_j}$$

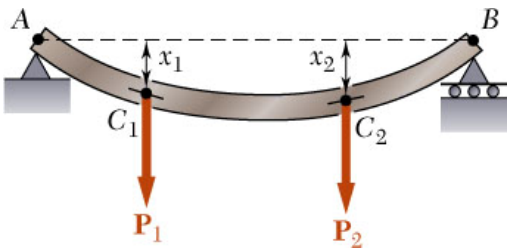
- ***U:*** *tüm kuvvetlerin şekil değiştirme enerjisi*

- Eğilmeye maruz kirişlerde;

$$U = \int_0^L \frac{M^2}{2EI} dx$$

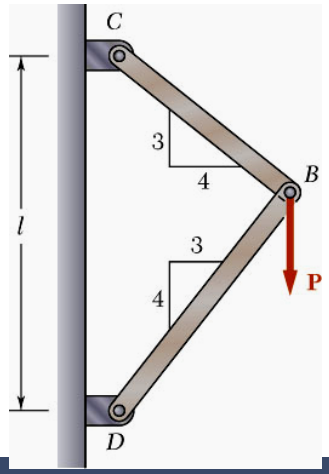
$$\boxed{x_j = \frac{\partial U}{\partial P_j} = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial P_j} dx}$$

$$\boxed{\theta_j = \frac{\partial U}{\partial P_j} = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial M_j} dx}$$



- Kafes sistemlerinde:

$$U = \sum_{i=1}^{n} \frac{F_i^2 L_i}{2 A_i E} \qquad x_j = \frac{\partial U}{\partial P_j} = \sum_{i=1}^{n} \frac{F_i L_i}{A_i E}\frac{\partial F_i}{\partial P_j}$$



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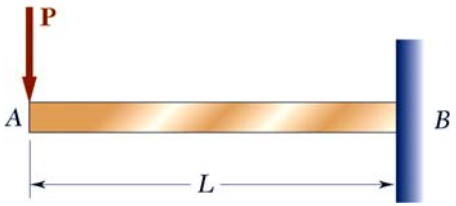
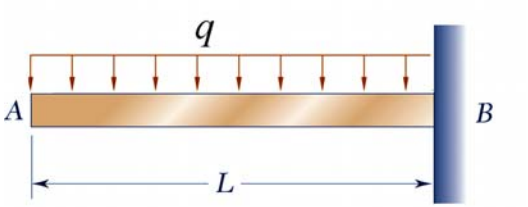
Örnek:

$y_A = ?$



Örnek:

$y_A = ?\ ,\ \ \theta_A = ?$



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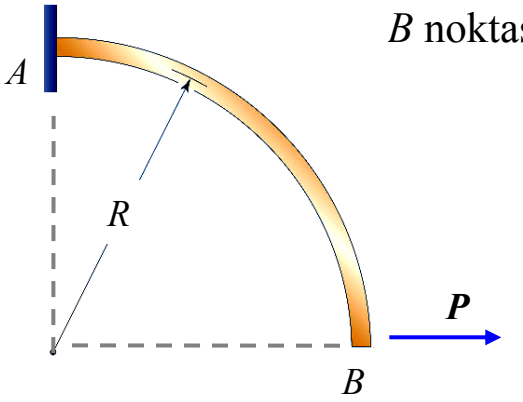
Örnek:



B noktasının yatay deplasmanını (x_B) ve eğilme açısını (θ_B) bulunuz.

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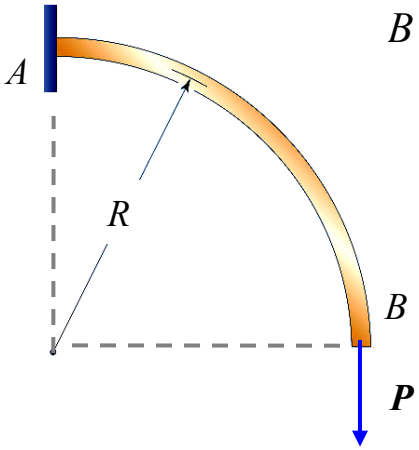
Örnek:



B noktasının yatay deplasmanını (y_B) ve eğilme açısını (θ_B) bulunuz.

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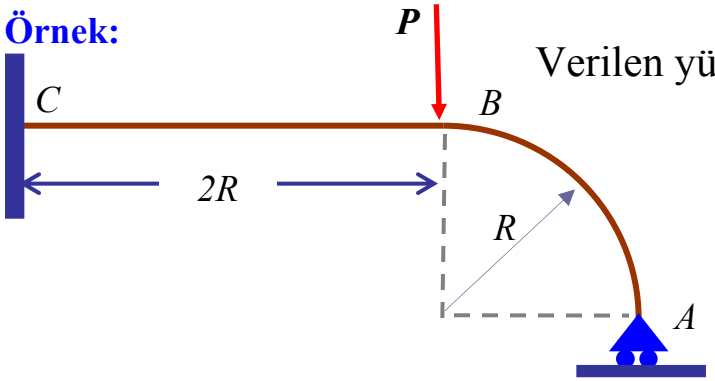
Örnek:



Verilen yükleme durumu için *A* noktasının reaksiyon kuvvetini bulunuz.

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